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THE STEADY MOTIONS OF A FLUID-FILLED SPHEROID ON A PLANE WITH FRICTION†

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The motion of a thin-walled spheroid completely filled with an ideal incompressible fluid executing uniform vortex motion (a Kelvin top) is investigated. It is assumed that the spheroid rests on a horizontal plane from which it is acted upon by a normal reaction and the force of viscous sliding friction. The equations of motion of a Kelvin top on a plane with friction are set up, and the conditions for them to be consistent are obtained. The steady and periodic motions of a Kelvin top are found, and problems of the stability and branching of these motions are investigated. © 2001 Elsevier Science Ltd. All rights reserved.

Fundamental results in solving a number of problems of solid-state dynamics with cavities containing fluid (the dynamics of fluid-filled gyroscopes, missiles and satellites) were obtained by Rumyantsev in [1]. Problems of the dynamics of rigid bodies with a fluid filling on a horizontal plane have been investigated to a lesser extent [2], although experiments by Kelvin with a thin-walled, fluid-filled spheroidal top are well known (see [1]). Below, the motion of a Kelvin top on a horizontal plane is investigated, taking into account viscous sliding friction at the point of contact of the top with the plane (unlike the formulation of the problem considered earlier [2], which assumed that there is neither friction nor slippage of the top at this point).

1. FORMULATION OF THE PROBLEM

Consider the motion of a thin-walled spheroid completely filled with an ideal incompresible fluid executing uniform vortex motion on horizontal plane, taking viscous sliding friction into account. It will be assumed that the mass of the spheroid walls is negligible compared with the mass of the fluid. Here, the centre of mass of the system and the principal central axes of inertia coincide with the centre S of the spheroid and its principal axes $Sx_1x_2x_3$ respectively.

Let a_1, $a_2 = a_1$ and a_3 be the semi-axis of the spheroid, $\delta = a_1/a_3$, ga_3 the acceleration due to gravity, $a_3 v_i$, ω_i, Ω_i and $\gamma_i (i = 1, 2, 3)$ the projections of the velocity of the centre of mass of the spheroid, the angular velocity, half the rotational vector and the unit vector of the rising vertical respectively onto the Sx_i axis $(i = 1, 2, 3)$, na_3 the magnitude of the normal reaction related to the mass of fluid, $\varkappa > 0$ the coefficient of viscous sliding friction and $r = \sqrt{\delta^2(\gamma_1^2 + \gamma_2^2) + \gamma_3^2}$ ($a_3 r$ is the distance from the centre of the spheroid to the supporting plane).

The equations of motion of the system, referred to the $Sx_1x_2x_3$ system of coordinates, have the form (cf. [1–4])

$$
\begin{aligned}
&\dot{v}_1 + \omega_2 v_3 - \omega_3 v_2 = (n-g)\gamma_1 - \varkappa[v_1 + (\delta^2\omega_3\gamma_2 - \omega_2\gamma_3)r^{-1}] \\
&\dot{v}_2 + \omega_3 v_1 - \omega_1 v_3 = (n-g)\gamma_2 - \varkappa[v_2 + (\omega_1\gamma_3 - \delta^2\omega_3\gamma_1)r^{-1}] \\
&\dot{v}_3 + \omega_1 v_2 - \omega_2 v_1 = (n-g)\gamma_3 - \varkappa[v_3 + \delta^2(\omega_2\gamma_1 - \omega_1\gamma_2)r^{-1}]'
\end{aligned}
\tag{1.1}
$$

$$
\frac{(\delta^2-1)^2}{5(\delta^2+1)}(\dot{\omega}_1 - \omega_2\omega_3) + \frac{4\delta^2}{5(\delta^2+1)}(\dot{\Omega}_1 - \Omega_2\omega_3) + \frac{2\delta^2}{5}\omega_2\Omega_3 = -(\delta^2-1)n\gamma_2\gamma_3 r^{-1} +
$$
$$
+\varkappa[(\delta^2 v_3\gamma_2 - v_2\gamma_3)r + \delta^4(\omega_2\gamma_1 - \omega_1\gamma_2)\gamma_2 - (\omega_1\gamma_3 - \delta^2\omega_3\gamma_1)\gamma_3]r^{-2}
$$

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$$\frac{(\delta^2-1)^2}{5(\delta^2+1)}(\dot\omega_2+\omega_1\omega_3)+\frac{4\delta^2}{5(\delta^2+1)}(\dot\Omega_2+\Omega_1\omega_3)-\frac{2\delta^2}{5}\omega_1\Omega_3=(\delta^2-1)n\gamma_1\gamma_3r^{-1}+$$
$$+\varkappa[-(\delta^2 v_3\gamma_1-v_1\gamma_3)r-\delta^4(\omega_2\gamma_1-\omega_1\gamma_2)\gamma_1-(\omega_2\gamma_3-\delta^2\omega_3\gamma_2)\gamma_3]r^{-2} \tag{1.2}$$

$$\frac{2}{5}\dot\Omega_3+\frac{4}{5(\delta^2+1)}(\omega_1\Omega_2-\omega_2\Omega_1)=\varkappa[(v_2\gamma_1-v_1\gamma_2)r+(\omega_1\gamma_1+\omega_2\gamma_2)\gamma_3-$$
$$-\delta^2\omega_3(\gamma_1^2+\gamma_2^2)]r^{-2}$$

$$\dot\Omega_1+\frac{2\delta^2}{\delta^2+1}(\omega_2-\Omega_2)\Omega_3-(\omega_3-\Omega_3)\Omega_2=0$$
$$\dot\Omega_2-\frac{2\delta^2}{\delta^2+1}(\omega_1-\Omega_1)\Omega_3+(\omega_3-\Omega_3)\Omega_1=0 \tag{1.3}$$
$$\dot\Omega_3+\frac{2}{\delta^2+1}(\omega_1\Omega_2-\omega_2\Omega_1)=0$$

$$\dot\gamma_1+\omega_2\gamma_3-\omega_3\gamma_2=0,\ \dot\gamma_2+\omega_3\gamma_1-\omega_1\gamma_3=0,\ \dot\gamma_3+\omega_1\gamma_2-\omega_2\gamma_1=0 \tag{1.4}$$

$$v_1\gamma_1+v_2\gamma_2+v_3\gamma_3+[(\delta^2\omega_3\gamma_2-\omega_2\gamma_3)+(\omega_1\gamma_3-\delta^2\omega_3\gamma_1)+\delta^2(\omega_2\gamma_1-\omega_1\gamma_2)]r^{-1}=0 \tag{1.5}$$

Systems (1.1) and (1.2) express theorems on the change in momentum and angular momentum of the spheroid respectively, system (1.3) expresses Helmholtz's theorem, system (1.4) expresses the condition that the unit vector of the rising vertical is constant in a fixed frame of reference and Eq. (1.5) expresses the condition of permanent contact of the spheroid with the plane during the motion. System (1.1)–(1.5) is closed with respect to the variables v_i, ω_i, Ω_i and γ_i ($i=1,2,3$) and n.

2. ANALYSIS OF THE EQUATIONS OF MOTION

First of all we note that system (1.1)–(1.5) contains no derivative of variable ω_3, which is due to the fact that we have neglected the mass of the spheroid and its symmetry about the Sx_3 axis. Therefore, the question of the consistency of system (1.1)–(1.5) requires further discussion. We will examine the third equations of systems (1.2) and (1.3). They are obviously consistent if, and only if, the following relation is satisfied

$$\delta^2\omega_3(\gamma_1^2+\gamma_2^2)=(\omega_1\gamma_1+\omega_2\gamma_2)\gamma_3+(v_2\gamma_1-v_1\gamma_2)r \tag{2.1}$$

Hence, the third equation of system (1.2) must be replaced by Eq. (2.1), which can be used to determine the variable ω_3, and using which this variable (in all cases when $\gamma_1^2+\gamma_2^2\neq 0$) can be eliminated from the remaining equations of system (1.1)–(1.5).

Then, multiplying the ith equation of system (1.1) by γ_i ($i=1,2,3$) and adding the relations obtained term by term, we have (taking Eq. (1.5) into account)

$$n=g+d(v_1\gamma_1+v_2\gamma_2+v_3\gamma_3)/dt \tag{2.2}$$

On the other hand, Eq. (1.5), using relation (2.1), can be represented in the form

$$v_1\gamma_1+v_2\gamma_2+v_3\gamma_3=(\delta^2-1)(\omega_1\gamma_2-\omega_2\gamma_1)\gamma_3/r \tag{2.3}$$

$$n=g+\ddot r\ \left(r=\sqrt{\delta^2(\gamma_1^2+\gamma_2^2)+\gamma_3^2}\equiv\sqrt{\delta^2(1-\gamma_3^2)+\gamma_3^2}\right) \tag{2.4}$$

Consequently, using relations (2.3) and (2.4), from Eqs (1.1) and (1.2) it is possible to eliminate the variables v_3 and n; here, Eq. (1.5) must be replaced by Eq. (2.3) which can be used to determine the variable v_3, and the third equation of system (1.1) must be discarded.

Thus, the motion of a thin-walled spheroid filled with an ideal incompressible fluid executing uniform vortex motion on a horizontal plane with friction can, generally speaking, be described by a system of differential equations comprising the first and second equations of (1.1) and the first and second equations of (1.2) (all of these, taking relations (2.1), (2.3) and (2.4) into account) and Eqs (1.3) and (1.4) (all of these taking relations (2.1) into account). This tenth-order system is used to determine the ten independent variables v_1, v_2, ω_1, ω_2, Ω_1, Ω_2, Ω_3, γ_1, γ_2 and γ_3, and allows of two first integrals (Helmholtz's and geometric)

$$\Omega_1^2+\Omega_2^2+\delta^2\Omega_3^2=\text{const},\quad \gamma_1^2+\gamma_2^2+\gamma_3^2=1 \tag{2.5}$$

The variables ω_3 and v_3 are determined from the finite Eqs (2.1) and (2.3) respectively, while relation (2.4) is used to determine the normal reaction.

3. STEADY AND PERIODIC MOTIONS

The above equations of motion of the spheroid obviously allow of steady motions of the form

$$\begin{gathered} v_1=v_2=v_3=\gamma_1=\gamma_2=\omega_1=\omega_2=\Omega_1=\Omega_2=0,\quad \gamma_3=\pm 1\\ \omega_3=\omega,\quad \Omega_3=\Omega \end{gathered} \tag{3.1}$$

(where ω and Ω are arbitrary constants; here $n = g$) and of the form

$$\begin{gathered} v_1=v_2=v_3=\gamma_3=\omega_3=\Omega_3=0,\quad \gamma_1=\alpha,\quad \gamma_2=\beta(\alpha^2+\beta^2=1)\\ \omega_1=\delta^2\omega\alpha,\quad \omega_2=\delta^2\omega\beta,\quad \Omega_1=\delta^2\Omega\alpha,\quad \Omega_2=\delta^2\Omega\beta \end{gathered} \tag{3.2}$$

(where ω and Ω are arbitrary constants, and α and β are arbitrary constants connected by the relation $\alpha^2 + \beta^2 = 1$; here, as before, $n = g$). These solutions correspond to permanent rotations of the spheroid about the vertically positioned axis of symmetry (solution (3.1)) or the vertically positioned diameter of the equatorial cross-section of the spheroid (solution (3.2)).

We will find the conditions for regular precessions of the spheroid to exist. These will be sought in the form

$$\begin{gathered} v_1=v_2=v_3=0,\quad \gamma_3=\gamma,\quad \omega_3=\omega\gamma,\quad \Omega_3=\Omega\gamma\\ \omega_1=\delta^2\omega\gamma_1,\quad \omega_2=\delta^2\omega\gamma_2,\quad \Omega_1=\mu\delta^2\Omega\gamma_1,\quad \Omega_2=\mu\delta^2\Omega\gamma_2 \end{gathered} \tag{3.3}$$

where γ, ω, Ω and μ are constants (here $n = g$).

Substituting relations (3.3) into the equations of motion of the spheroid and assuming $\Omega = k\omega$, we conclude that γ_1 and γ_2 must satisfy the system of two differential equations

$$\dot\gamma_1=-\omega(\delta^2-1)\gamma\gamma_2,\quad \dot\gamma_2=\omega(\delta^2-1)\gamma\gamma_1 \tag{3.4}$$

and the four constants μ, ω, γ and $k = \Omega/\omega$ must satisfy the two finite relations

$$\mu[k(\delta^2-1)+\delta^2(\delta^2+1)]=2\delta^2 \tag{3.5}$$

$$2k^2+k(\delta^2-1)(1+F)+\delta^2[(\delta^2+1)F-(\delta^2-1)]=0 \tag{3.6}$$

where

$$F=F(\omega,\gamma)=\frac{5g}{\omega^2\delta^4\sqrt{\delta^2(1-\gamma^2)+\gamma^2}} \tag{3.7}$$

It follows from Eqs (3.4) that γ_1 and γ_2 are periodic functions of time (with $\omega\gamma \neq 0$), and here (see (3.3)) ω_1, ω_2, Ω_1 and Ω_2 are also periodic functions of time. The constant μ is determined from Eq. (3.5), while the constant k is determined from Eq. (3.6). Thus, regular precessions exist if Eq. (3.6) is solvable, and they form two-parameter families (the free parameters $\omega \in \mathbf{R}$ and $\gamma \in [-1; 1]$).

When $\gamma = \pm 1$ or $\gamma = 0$, solutions (3.3) are converted into permanent rotations of the spheroid about the vertically positioned axis of symmetry (3.1) or about the vertically positioned diameter of the equatorial cross-section of the spheroid (3.2) respectively.

The condition for regular precessions (3.3) to exist (the condition for Eq. (3.6) to be solvable) has the form

$$(\delta^2-1)^2F^2-2(3\delta^4+6\delta^2-1)F+(\delta^2-1)(9\delta^2-1)\geqslant 0 \tag{3.8}$$

and, generally speaking, imposes a constraint on the parameters ω and γ (see (3.7)).
Inequality (3.8) occurs if $F \leqslant F_1$ or $F \geqslant F_2$, where

$$F_1=\frac{(\delta-1)(3\delta+1)}{(\delta+1)^2},\quad F_2=\frac{(\delta+1)(3\delta-1)}{(\delta-1)^2}$$

Since $F > 0$ (see (3.7)), the inequality $F \leqslant F_1$ can be satisfied only when $\delta > 1$, while inequality $F \geqslant F_2$ is always satisfied when $3\delta \leqslant 1$.

Note that steady and periodic motions (3.1)–(3.3) are of greatest interest in the case when $0 < \Omega/\omega \leqslant 1$, i.e. in the case when $0 < k \leqslant 1$. If $0 < k \ll 1$, Eq. (3.6) can be represented in the form

$$\frac{5g}{\omega^2\delta^4\sqrt{\delta^2(1-\gamma^2)+\gamma^2}}=\frac{\delta^2-1}{\delta^2+1}\left(1-\frac{2}{\delta^2+1}k\right)+o(k) \tag{3.9}$$

Here, regular precessions exist only when $\delta > 1$. If $k = 1$, Eq. (3.6) can be written in the form

$$\frac{5g}{\omega^2\delta^4\sqrt{\delta^2(1-\gamma^2)+\gamma^2}}=\frac{\delta^4-2\delta^2-1}{\delta^4+2\delta^2-1} \tag{3.10}$$

Here, regular precessions exits if $\delta^2 \in (0, \sqrt{2}-1) \cup (\sqrt{2}+1, +\infty)$.

4. THE STABILITY OF PERMANENT ROTATIONS OF THE SPHEROID

We will consider steady motion (3.1) and write, in its neighbourhood, linearized equations of perturbed motion, assuming

$$\omega_3=\omega+\omega_3',\quad \Omega_3=\Omega+\Omega_3',\quad \gamma_3=1+\gamma_3',\quad n=g+n'$$

and retaining the previous notation for the remaining variables $v_i (i = 1, 2, 3)$, ω_j, Ω_j and $\gamma_j (j = 1, 2)$. Eliminating the variables v_3, ω_3', Ω_3', γ_3' and n', using relations (2.1), (2.3), (2.4) and (2.5), after simple but rather lengthy calculations we obtain

$$\dot\gamma_1-\omega\gamma_2+\omega_2=0 \tag{4.1}$$

$$\dot\gamma_2+\omega\gamma_1-\omega_1=0$$

$$\dot\Omega_1-\omega\Omega_2-\frac{\delta^2-1}{\delta^2+1}\Omega\Omega_2+\frac{2\delta^2}{\delta^2+1}\Omega\omega_2=0 \tag{4.2}$$

$$\dot\Omega_2+\omega\Omega_1+\frac{\delta^2-1}{\delta^2+1}\Omega\Omega_1-\frac{2\delta^2}{\delta^2+1}\Omega\omega_1=0$$

$$\dot{v}_1 - \omega v_2 + \varkappa v_1 + \varkappa\omega\delta^2\gamma_2 - \varkappa\omega_2 = 0 \tag{4.3}$$

$$\dot{v}_2 + \omega v_1 + \varkappa v_2 - \varkappa\omega\delta^2\gamma_1 + \varkappa\omega_1 = 0$$

$$\dot{\omega}_1 - \omega\omega_2 + 5\varkappa\frac{\delta^2+1}{(\delta^2-1)^2}\omega_1 + \frac{2\delta^2}{\delta^2+1}\Omega\omega_2 - 5\varkappa\frac{\delta^2(\delta^2+1)}{(\delta^2-1)^2}\omega\gamma_1 +$$

$$+5g\frac{\delta^2+1}{\delta^2-1}\gamma_2 + \frac{4\delta^2}{\delta^4-1}\Omega\Omega_2 + 5\varkappa\frac{\delta^2+1}{(\delta^2-1)^2}v_2 = 0 \tag{4.4}$$

$$\dot{\omega}_2 + \omega\omega_1 + 5\varkappa\frac{\delta^2+1}{(\delta^2-1)^2}\omega_2 - \frac{2\delta^2}{\delta^2+1}\Omega\omega_1 - 5\varkappa\frac{\delta^2(\delta^2+1)}{(\delta^2-1)^2}\omega\gamma_2 -$$

$$-5g\frac{\delta^2+1}{\delta^2-1}\gamma_1 - \frac{4\delta^2}{\delta^4-1}\Omega\Omega_1 - 5\varkappa\frac{\delta^2+1}{(\delta^2-1)^2}v_1 = 0$$

Assuming now that

$$x = (\gamma_1 + i\gamma_2)e^{i\omega t}, \quad y = (\Omega_1 + i\Omega_2)e^{i\omega t} \tag{4.5}$$

$$v = (v_1 + iv_2)e^{i\omega t}, \quad w = (\omega_1 + i\omega_2)e^{i\omega t}$$

we reduce the eighth-order system (4.1)–(4.4) in eight real variables to a fourth-order system in four complex variables x, y, v and w

$$\dot{x} - iw = 0$$

$$\dot{y} + i\frac{\delta^2-1}{\delta^2+1}\Omega y - 2i\frac{\delta^2}{\delta^2+1}\Omega w = 0$$

$$\dot{v} + \varkappa v - i\varkappa\omega\delta^2 x + i\varkappa w = 0 \tag{4.6}$$

$$\dot{w} + \left[5\varkappa\frac{\delta^2+1}{(\delta^2-1)^2} - 2i\frac{\delta^2}{\delta^2+1}\Omega\right]w - 5\frac{\delta^2+1}{(\delta^2-1)^2}[\varkappa\omega\delta^2 + i(\delta^2-1)g]x -$$

$$-4i\frac{\delta^2}{\delta^4-1}\Omega y + 5i\varkappa\frac{\delta^2+1}{(\delta^2-1)^2}v = 0$$

The characteristic equation for system (4.6) has the form

$$f(\lambda) \equiv \lambda^4 + (p_1 + iq_1)\lambda^3 + (p_2 + iq_2)\lambda^2 + (p_3 + iq_3)\lambda + iq_4 = 0 \tag{4.7}$$

$$p_1 = \frac{\delta^4 + 3\delta^2 + 6}{(\delta^2-1)^2}\varkappa, \qquad p_2 = \frac{5(\delta^2+1)g + 2\delta^2\Omega^2}{\delta^2-1}$$

$$p_3 = \frac{5(\delta^2+1)g + 2\delta^2\Omega^2 + 5\delta^2\Omega\omega}{\delta^2-1}\varkappa$$

$$q_1 = -\Omega, \quad q_2 = -\frac{5\delta^2(\delta^2+1)\omega + (\delta^4 - 7\delta^2 + 6)\Omega}{(\delta^2-1)^2}\varkappa$$

$$q_3 = 5\Omega g, \quad q_4 = 5\Omega g\varkappa$$

Obviously, permanent rotation (3.1) is stable, asymptotically with respect to all the variables, except, generally speaking, the variable Ω_3, if all roots of Eq. (4.7) lie in the left-hand half-plane (see relations (4.5), (2.1), (2.3) and (2.5)), and unstable if at least one root of this equation lies in the right-hand half-plane.

All roots of Eq. (4.7) have negative real parts if, and only if, the Juri matrix

$$\begin{Vmatrix} 1 & -q_1 & -p_2 & q_3 & 0 & 0 & 0 \\ 0 & 1 & -q_1 & -p_2 & q_3 & 0 & 0 \\ 0 & 0 & 1 & -q_1 & -p_2 & q_3 & 0 \\ 0 & 0 & 0 & p_1 & -q_2 & -p_3 & q_4 \\ 0 & 0 & p_1 & -q_2 & -p_3 & q_4 & 0 \\ 0 & p_1 & -q_2 & -p_3 & q_4 & 0 & 0 \\ p_1 & -q_2 & -p_3 & q_4 & 0 & 0 & 0 \end{Vmatrix} \tag{4.8}$$

is innerly positive [5]. The conditions of inner positivity of matrix (4.8) have the form of three inequalities, which impose constraints on the parameters of the spheroid (i.e. on δ) and on the parameters of permanent rotation (3.1) (i.e. on ω and Ω). They do not depend on the coefficient of friction $\varkappa \in (0; +\infty)$.

5. ANALYSIS OF SPECIAL CASES

In the general case, the conditions of inner positivity of matrix (4.8) are extremely cumbersome, so we will therefore confine ourselves to an investigation of the special cases of weakly vortex ($0 < \Omega/\omega \ll 1$) and solid-state ($\Omega = \omega$) motions of the fluid.

If $0 < \Omega/\omega \ll 1$, all roots of Eq. (4.7) lie in the left-hand half-plane if, and only if

$$(\delta^2-1)[g(\delta^4+3\delta^2+6+o(1))-\delta^4\omega^2(\delta^2-1)]>0 \tag{5.1}$$

$$(\delta^2-1)[5g(\delta^2+1+o(1))-\delta^4\omega^2(\delta^2-1)]>0 \tag{5.2}$$

Consequently, permanent rotations of a spheroid filled with a fluid executing weakly vortex motion about a vertically positioned axis of symmetry (solutions (3.1) with $0 < \Omega/\omega \ll 1$) are stable if the spheroid is oblate along the axis of symmetry ($\delta > 1$) and its angular velocity is fairly low ($\omega^2 < \omega_{01}^2$). Here (see 5.1) and (5.2))

$$\omega_{01} = \sqrt{\frac{5g(\delta^2+1)}{\delta^4(\delta^2-1)}(1+o(1))} \tag{5.3}$$

For the critical value (5.3) of the angular velocity, there is a loss of stability of permanent rotations (3.1), for which $0 < \Omega/\omega \ll 1$, due to the production of regular precessions (3.3) for which $0 < k \ll 1$. These precessions exit only when $\delta > 1$ and $\omega_{00}^2 < \omega^2 < \omega_{01}^2$, where $\omega_{00} = \omega_{01}/\sqrt{\delta}$.

When $\omega^2 = \omega_{00}^2$, regular precessions (3.3) ($0 < k \ll 1$) are converted into permanent rotations of the spheroid about the vertically positioned diameter of its equatorial cross-section (3.2) ($0 < \Omega/\omega \ll 1$) which are stable (according to bifurcation theory [6]) at a fairly high angular velocity ($\omega^2 > \omega_{00}^2$) if $\delta > 1$, and also at any angular velocity if $\delta < 1$.

When $\Omega = \omega$, all the roots of Eq. (4.7) lie in the left-hand half-plane if, and only if

$$(\delta^2-1)[5g(\delta^2+1)^2(\delta^4+3\delta^2+6)-\omega^2\delta^4(7\delta^6-15\delta^4+9\delta^2-41)]>0 \tag{5.4}$$

$$\begin{aligned}&(\delta^2-1)[125g^3(\delta^2+1)^5-25g^2\omega^2\delta^4(\delta^2+1)(\delta^6-\delta^4-\delta^2-23)-\\&-40g\omega^4\delta^4(2\delta^8-3\delta^6-10\delta^4-17\delta^2+6)-56\omega^6\delta^8(\delta^4-2\delta^2-1)]>0\end{aligned} \tag{5.5}$$

$$(\delta^2-1)[5g(\delta^4+2\delta^2-1)-\omega^2\delta^4(\delta^4-2\delta^2-1)]>0 \tag{5.6}$$

Consequently, permanent rotations of a spheroid filled with a fluid executing uniform vortex motion, during which the fluid and shell rotate, as a whole about a vertically positioned axis of symmetry, are stable only if the spheroid is oblate along the axis of symmetry ($\delta > 1$). Here, solutions (3.1), for which $\Omega = \omega$, are stable at any angular velocity in the case of a slightly oblate spheroid ($\delta^2 \leqslant \sqrt{2} + 1$) and at a fairly low angular velocity ($\omega^2 < \omega_{11}^2$) in the case of an extremely oblate spheroid ($\delta^2 > \sqrt{2} + 1$) (see (5.4)–(5.6)). Here

$$\omega_{11} = \sqrt{\frac{5g(\delta^4 + 2\delta^2 - 1)}{\delta^4(\delta^4 - 2\delta^2 - 1)}} \tag{5.7}$$

With the critical value (5.7) of the angular velocity, there is a loss of stability of permanent rotations (3.1) ($\Omega = \omega$) of an extremely oblate spheroid ($\delta^2 > \sqrt{2} + 1$) due to the production of regular precessions (3.3) ($k = 1$). These precessions exits if $\delta^2 > \sqrt{2} + 1$ only when $\omega_{10}^2 < \omega^2 < \omega_{11}^2$, where $\omega_{10} = \omega_{11}/\sqrt{\delta}$.

When $\omega^2 = \omega_{10}^2$, the regular precessions (3.3) ($k = 1$) are converted into permanent rotations about the vertically positioned diameter of its equatorial cross-section (3.2) ($\Omega = \omega$) which are stable (according to bifurcation theory) when $\delta^2 > \sqrt{2} + 1$ and $\omega^2 > \omega_{10}^2$.

Remark. Regular precessions (3.3) for which $k = 1$ ($\Omega = \omega$) also exist (see Section 3) in the case of an extremely prolate spheroid ($\delta^2 < \sqrt{2} - 1$) if $\omega_{11}^2 < \omega^2 < \omega_{10}^2$. At ω_{11}^2 and ω_{10}^2, these precessions are converted into permanent rotations (3.1) or (3.2) ($\Omega = \omega$) respectively. When $\omega^2 = \omega_{11}^2$ there is a change in the number of roots of Eq. (4.7) with a positive real part, that always exist if $\delta < 1$ (see relations (5.4)–(5.6)). It can be shown that ω_{10}^2 is the critical value of the angular velocity of permanent rotations (3.2) ($\Omega = \omega$) of an extremely prolate ($\delta^2 < \sqrt{2} - 1$) spheroid: when $\omega^2 < \omega_{00}^2$ these rotations are stable, but when $\omega^2 > \omega_{10}^2$ they are unstable. Permanent rotations (3.2) ($\Omega = \omega$) of a slightly prolate. Spheroid ($\sqrt{2} - 1 \leqslant \delta^2 < 1$) are stable at any angular velocity.

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REFERENCES

1. MOISEYEV, N. N. and RUMYANTSEV, V. V., *The Dynamics of a Body with Cavities Containing a Fluid*. Nauka, Moscow, 1965.
2. MARKEYEV, A. P., The stability of the rotation of a top with a cavity filled with a fluid. *Izv. Akad. Nauk SSSR. MTT*, 1985, 3, 19–26.
3. KARAPETYAN, A. V., *The Stability of Steady Motions*. Editorial URSS, Moscow, 1998.
4. KARAPETYAN, A. V. and PROKONINA, O. V., The stability of permanent rotations of a top with a cavity filled with a fluid on a plane with friction. *Prikl. Mat. Mekh.*, 2000, **64**, 1, 85–91.
5. JURI, E. I., *Inners and Stability of Dynamical Systems*. Wiley, New York, 1974.
6. MARSDEN, J. E. and McCRACKEN, M., *The Hopf Bifurcation and its Applications*. Springer, New York, 1976.

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